Lesson 3-5 The Graph Scale-Change Theorem

Vocabulary

horizontal and vertical scale change, scale factor

size change

▶ **BIG IDEA** The graph of a function can be scaled horizontally, vertically, or in both directions at the same time.

Vertical Scale Changes

Consider the graph of $y = f1(x) = x^3 + 3x^2 - 4x$ shown both in the graph and function table below. What happens when you multiply all the *y*-values of the graph by 2? What would the resulting graph and table look like? Activity 1 will help you answer these questions.

Mental Math

What is $R_{270°}(1, 0)$?

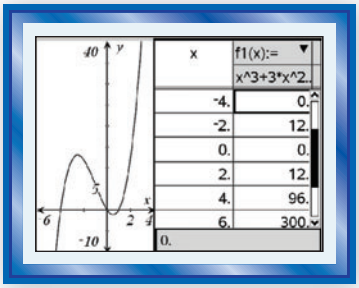
Activity 1

MATERIALS Graphing utility or slider graph application from your teacher

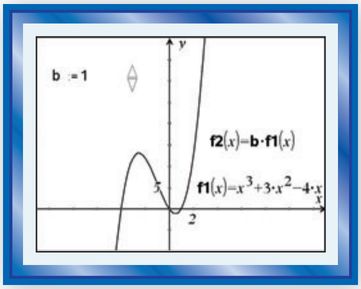
Step 1 Graph $f1(x) = x^3 + 3x^2 - 4x$ with window $-11 \le x \le 13$ and $-10 \le y \le 40$.

Step 2 Graph $f2(x) = 3(x^3 + 3x^2 - 4x) = 3 \cdot f1(x)$ on the same axes. Fill in the table of values for $f1(x)$ and $f2(x)$ only. Describe how the $f2(x)$ values relate to the $f1(x)$ values.

x	*f1*(*x*)	*f2*(*x*)	0.5 • *f1*(*x*)	1.5 • *f1*(*x*)	2 • *f1*(*x*)
–4	?	?	?	?	?
2	?	?	?	?	?
7	?	?	?	?	?
10	?	?	?	?	?



Step 3 Repeat Step 2 with $f3(x) = b(x^3 + 3x^2 - 4x)$ and use a slider to vary the value of *b*. Set the slider to 0.5, 1.5, and then 2. For each of these *b*-values, use a function table to fill in a column of the table in Step 2. Describe how the $f3(x)$ values relate to the corresponding values of $f1(x)$.



In Step 2 of Activity 1, we say that the graph of $f2$ is the image of the graph of $f1$ under a *vertical scale change* of magnitude 3. Each point on the graph of $f2$ is the image of a point on the graph of f under the mapping $(x, y) \to (x, 3y)$. You can create the same change by replacing y with $\frac{y}{3}$ in the equation for $f1$, because $\frac{y}{3} = x^3 + 3x^2 - 4x$ is equivalent to $y = 3(x^3 + 3x^2 - 4x)$, or $y = 3f1(x)$ in function notation. Similarly, if you replace y with $2y$ in the original equation, you obtain $2y = x^3 + 3x^2 - 4x$, which is equivalent to $y = \frac{1}{2}(x^3 + 3x^2 - 4x)$ or $y = \frac{1}{2}f(x)$.

STOP QY

Horizontal Scale Changes

Replacing the variable y by ky in an equation results in a vertical scale change. What happens when the variable x is replaced by $\frac{x}{2}$? By $4x$?

> **QY**
>
> Replacing y by $\frac{y}{4}$ in the equation for $f1$ yields an equation equivalent to $y =$ __?__. What is the effect on the graph of f?

Activity 2

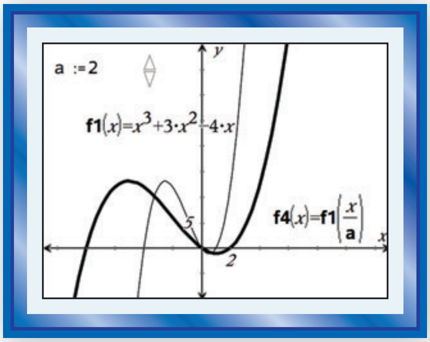
MATERIALS Graphing utility or slider graph application provided by your teacher.

Step 1 Consider the graph of $f1$ in Activity 1. Complete the table at the right.

x	$f1(x)$
–4	?
–1	?
2	?

Step 2 If $f4(x) = f1\left(\frac{x}{a}\right)$, then $f4(x) = \left(\frac{x}{a}\right)^3 + 3\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right)$. Graph $f4$ and use a slider to vary the value of a.

Step 3 **a.** What are the x- and y-intercepts of $f1$?
b. How do the intercepts of $f4$ change as a changes?



The graph of $f4$ in Activity 2 is the image of the graph of $f1$ under a *horizontal scale change* of magnitude a. Each point on the graph of $f4$ is the image of a point on the graph of $f1$ under the mapping $(x, y) \rightarrow (ax, y)$. Replacing x by $\frac{x}{2}$ in the equation doubles the x-values of the preimage points while the corresponding y-values remain the same. Accordingly, the x-intercepts of the image are two times as far from the y-axis as the x-intercepts of the preimage.

The Graph Scale-Change Theorem

In general, a **scale change** centered at the origin with **horizontal scale factor** $a \neq 0$ and **vertical scale factor** $b \neq 0$ is a transformation that maps (x, y) to (ax, by). The scale change S can be described by

$$S: (x, y) \rightarrow (ax, by) \quad \text{or} \quad S(x, y) = (ax, by).$$

If $a = 1$ and $b \neq 1$, then the scale change is a **vertical scale change**. If $b = 1$ and $a \neq 1$, then the scale change is a **horizontal scale change**. When $a = b$, the scale change is called a **size change**. Notice that in the preceding instances, replacing x by $\frac{x}{2}$ in an equation for a function results in the scale change $S: (x, y) \rightarrow (2x, y)$; and replacing y by $\frac{y}{3}$ leads to the scale change $S: (x, y) \rightarrow (x, 3y)$. These results generalize.

Graph Scale-Change Theorem

Given a preimage graph described by a sentence in x and y, the following two processes yield the same image graph:

(1) replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$ in the sentence;

(2) applying the scale change $(x, y) \rightarrow (ax, by)$ to the preimage graph.

Proof Name the image point (x', y'). So $x' = ax$ and $y' = by$. Solving for x and y gives $\frac{x'}{a} = x$ and $\frac{y'}{b} = y$. The image of $y = f(x)$ will be $\frac{y'}{a} = f\left(\frac{x'}{a}\right)$. The image equation is written without the primes.

Unlike translations, scale changes do not produce congruent images unless $a = b = 1$. Notice also that multiplication in the scale change corresponds to division in the equation of the image. This is analogous to the Graph-Translation Theorem in Lesson 3-2, where addition in the translation $(x, y) \rightarrow (x + h, y + k)$ corresponds to subtraction in the image equation $y - k = f(x - h)$.

GUIDED

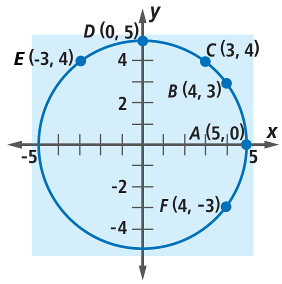
Example 1

The relation described by $x^2 + y^2 = 25$ is graphed at the right.

a. Find images of points labeled A–F on the graph under S: $(x, y) \rightarrow (2x, y)$.

b. Copy the circle onto graph paper; then graph the image on the same axes.

c. Write an equation for the image relation.



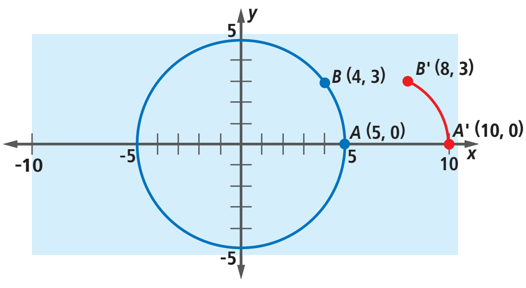
Solution

a. Copy and complete the table below.

b. Plot the preimage and image points on graph paper and draw a smooth curve connecting the image points. A partial graph is drawn below.

c. According to the Graph Scale-Change Theorem, an equation for an image under S: $(x, y) \rightarrow (2x, y)$ can be found by replacing x by __?__ in the equation for the preimage. The result is the equation __?__ .

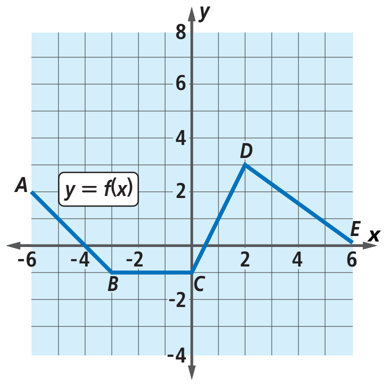
	Preimage		Image	
Point	x	y	$2x$	y
A	5	0	10	0
B	?	?	?	?
C	?	?	?	?
D	?	?	?	?
E	?	?	?	?
F	?	?	?	?



Example 2

A graph and table for $y = f(x)$ are given at the right. Draw the graph of $\frac{y}{3} = f(2x)$.

x	$f(x)$
–6	2
–3	–1
0	–1
2	3
6	0



Solution Rewrite $\frac{y}{3} = f(2x)$ as $\frac{y}{3} = f\left(\frac{x}{\frac{1}{2}}\right)$. By the Graph Scale-Change Theorem, replacing x by $\frac{x}{\frac{1}{2}}$ and y by $\frac{y}{3}$ is the same as applying the scale change $(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$.

So, $A = (-6, 2) \Rightarrow (-3, 6) = A'$

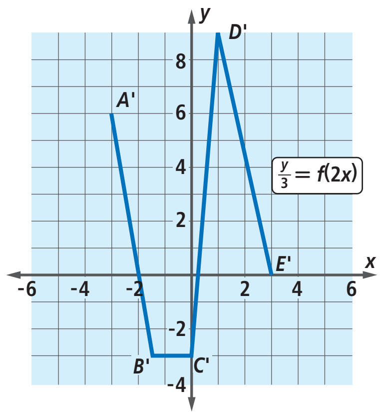
$B = (-3, -1) \Rightarrow (-3/2, -3) = B'$

$C = (0, -1) \Rightarrow (0, -3) = C'$

$D = (2, 3) \Rightarrow (1, 9) = D'$

and $E = (6, 0) \Rightarrow (3, 0) = E'$.

The graph of the image is shown at the right.



Negative Scale Factors

Notice what happens when a scale factor is –1. Consider the horizontal and vertical scale changes H and V with scale factors equal to –1.

$$H: (x, y) \rightarrow (-x, y) \text{ and } V: (x, y) \rightarrow (x, -y)$$

In H, each x-value is replaced by its opposite, which produces a reflection over the y-axis. Similarly, in V, replacing y by $-y$ produces a reflection over the x-axis. More generally, a scale factor of $-k$ combines the effect of a scale factor of k and a reflection over the appropriate axis.

Questions

COVERING THE IDEAS

1. **True or False** Under every scale change, the preimage and image are congruent.

2. Under a scale change with horizontal scale factor a and vertical factor b, the image of (x, y) is __?__.

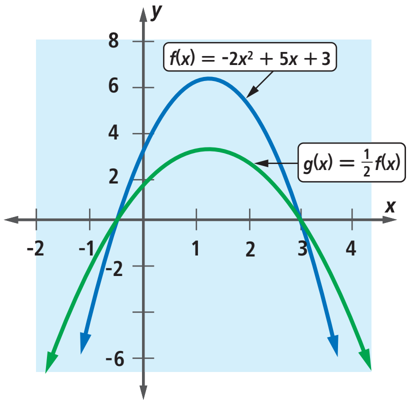
3. Refer to the Graph Scale-Change Theorem. Why are the restrictions $a \neq 0$ and $b \neq 0$ necessary?

4. If S maps each point (x, y) to $\left(\frac{x}{2}, 6y\right)$, give an equation for the image of $y = f(x)$ under S.

5. Consider the function $f1$ used in Activities 1 and 2.
   a. Write a formula for $f1\left(\frac{x}{3}\right)$.
   b. How is the graph of $y = f1\left(\frac{x}{3}\right)$ related to the graph of $y = f1(x)$?

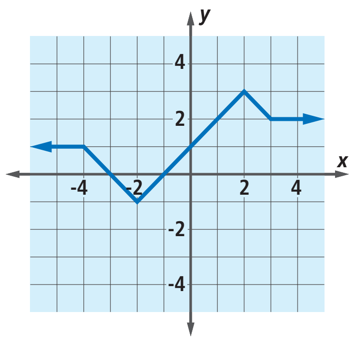
6. **Multiple Choice** Which of these transformations is a size change?
   A $(x, y) \rightarrow (3x, 3y)$ B $(x, y) \rightarrow (3x, y)$
   C $(x, y) \rightarrow (x + 3, y + 3)$ D $(x, y) \rightarrow \left(\frac{x}{3}, y\right)$

7. Functions f and g with $f(x) = -2x^2 + 5x + 3$ and $g(x) = \frac{1}{2}f(x)$ are graphed at the right.
   a. What scale change maps the graph of f to the graph of g?
   b. The x-intercepts of f are at $x = -\frac{1}{2}$ and $x = 3$. Where are the x-intercepts of g?
   c. How do the y-intercepts of f and g compare?
   d. The vertex of the graph of f is (1.25, 6.125). What is the vertex of the graph of g?



8. Consider the parabola with equation $y = x^2$. Let $S(x, y) = \left(2x, \frac{y}{7}\right)$.
   a. Find the images of (-3, 9), (0, 0), and $\left(\frac{1}{2}, \frac{1}{4}\right)$ under S.
   b. Write an equation for the image of the parabola under S.

9. The graph of a function f is shown at the right.
   a. Graph the image of f under $S(x, y) = \left(\frac{1}{2}x, 3y\right)$.
   b. Find the x- and y-intercepts of the image.
   c. Find the coordinates of the point where the y-value of the image of f reaches its maximum.

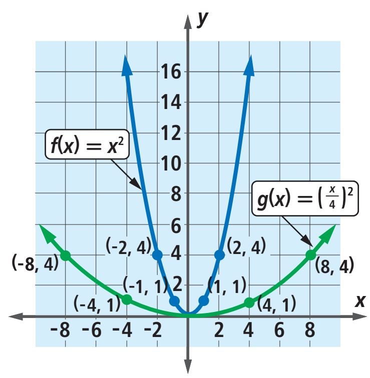


10. Give another name for the horizontal scale change of magnitude -1.

11. Describe the scale change that maps the graph of $y = \sqrt{x}$ onto the graph of $y = \sqrt{\frac{x}{12}}$.

APPLYING THE MATHEMATICS

12. Refer to the parabolas at the right. The graph of g is the image of the graph of f under what
   a. horizontal scale change? b. vertical scale change?
   c. size change?

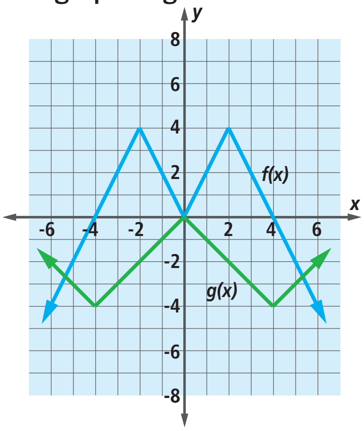


13. Write an equation for the image of the graph of $y = x + \frac{1}{x}$ under each transformation.
   a. $S(x, y) = (2x, 2y)$ b. $S(x, y) = \left(\frac{x}{3}, -y\right)$

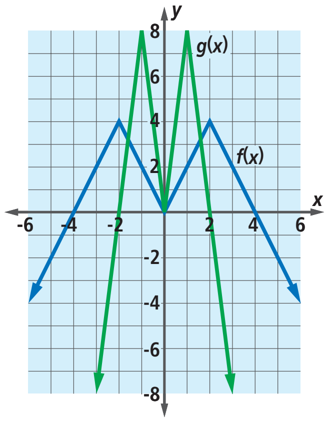
14. A scale change maps (10, 0) onto (2, 0) and (-5, 8) onto (-1, 2). What is the equation of the image of the graph of $f(x) = x^3 - 8$ under the scale change?

In 15 and 16, give a rule for a scale change that maps the graph of f onto the graph of g.

15.



16.



REVIEW

In 17 and 18, an equation for a function is given. Is the function odd, even, or neither? If the function is odd or even, prove it. (Lesson 3-4)

17. $f(x) = (5x + 4)^3$

18. $g(x) = 5x^4 + 4$

19. If $f(x) = -g(x)$ for all x in the common domain of f and g, how are the graphs of f and g related? **(Lesson 3-4)**

20. One of the parent functions presented in Lesson 3-1 has a graph that is not symmetric to the x-axis, y-axis, or origin. It has the asymptote $y = 0$. Which is it? **(Lesson 3-1)**

21. The table at the right shows the number of injuries on different types of rides in amusement parks in the U.S. in 2003, 2004, and 2005. Use the table to explain whether each statement is supported by the data. **(Lesson 1-1)**

	2003	2004	2005
Total	1954	1648	1713
Children's Rides	277	219	192
Family and Adult Rides	1173	806	1131
Roller Coasters	504	613	390

Source: National Safety Council

a. Injuries on children's rides decreased slightly each year.

b. The number of injuries on roller coasters decreased from 2003 to 2004.

c. Roller coasters are not as safe as children's rides.

22. Pizza π restaurant made 300 pizzas yesterday; 64% of the pizzas had no toppings, 10% of the pizzas had two toppings, and 26% had more than two toppings. How many pizzas had at least two toppings? **(Previous Course)**

EXPLORATION

23. a. Explore $g(x) = b(x^3 + 3x^2 - 4x)$ for $b < 0$. Explain what happens to the graph of g as b changes.

b. Explore $h(x) = \left(\frac{x}{a}\right)^3 + 3\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right)$ for $a < 0$. Explain what happens to the graph of h as a changes.

QY ANSWER

$4(x^3 + 3x^2 - 4x)$; a vertical scale change of magnitude 4